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Table of the First Moment of Ranked Extremes

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Let a sample in n independent random values from the extreme-value distribution, with c.d.f. $\Phi(y)=\exp(-e^{-y})$, be arranged in decreasing order and denoted by $y_1, y_2, \ldots, y_m, \ldots, y_n$. The table gives the expected values for all these order statistics for sample size not exceeding 25. For the larger samples, up to $n=100$, the expected values are given only for the first 26 largest values.

1. Explanation and Use of Table

Table 1, prepared at the suggestion of Bradford F. Kimball, gives the expected values, $E(y_m)$, of order statistics from the type I distribution of "reduced" largest values, Prob$\{Y\leq y\}=\Phi(y)=\exp(-e^{-y})$. This distribution was discussed in considerable detail in a series of lectures by E. J. Gumbel [1].[1] The entries were computed from the formula given by B. F. Kimball [2],

$$E(y_m)=C+\sum_{t=0}^{m-1}(-1)^t\binom{n}{t}\Delta^t \ln(n-t), \qquad m=1,2, \ldots, n, \quad (1)$$

where

$$y_1\geq y_2\geq \ldots \geq y_m\geq \ldots \geq y_{n-1}\geq y_n$$

denotes the order statistics, that is, the observations in a sample of n arranged in *decreasing* order of magnitude. The quantity $C=0.57721566\ldots$ is Euler's constant. The values $E(y_m)$ are tabulated herein for

$$m=1(1)\min(n,26)$$

$$n=1(1)10(5)60(10)100.$$

The table of the first moment of ranked extremes is one of the many tables furnishing pertinent information concerning the extreme-value distribution of type I. At one time Kimball [2] advocated use of such values in plotting extreme-value data on a special-scale extreme-value plotting paper. Because in this method the *largest* sample values are important, computations have not been carried beyond the 26th largest value ($m=26$) for sample sizes $n>26$. Subsequent investigation [3] indicated other criteria that should be given consideration, such as the shortest confidence interval of fixed probability, which applies to the plotting position.

However, there arise other situations in which it is important to know where these mean values $E(y_m)$ are located, and with the great increase in application of the extreme-value distribution this table should be a valuable asset.

A recent important use relates to the determination of best linear unbiased estimators for the parameters u, β of the distribution of largest values [2] in general form

$$\text{Prob}\ \{X\leq x\}=F(x)=\exp(-e^{-y}), \qquad y=(x-u)/\beta,$$

or in general, for estimators of a given linear combination such as $u+a\beta$ with a known. The condition of unbiasedness involves a knowledge of the means, $E(y_m)$, of the order statistics in samples from the reduced distribution $\Phi(y)$. These are provided by the present table. The second condition, "best" i. e., minimum variance subject to the first condition, involves in addition the variances and covariances of the order statistics $\sigma^2(y_m)$, $\sigma(y_my_p)$.

This problem has been treated by Lieblein [4], and numerical results up to $n=6$ appear in a paper by Lieblein and Zelen [5].

2. Interpolation Chart [3] for $E(y_m)$

In order to furnish values of $E(y_m)$ for samples of sizes not included in the table an interpolation chart designed by B. F. Kimball is appended (fig. 1).

The curves in the chart connect points having the same ranking. Thus, the lowest curve is for the expected values of the largest in a sample of size n; the second lowest curve relates to the second largest, etc. Each curve has two branches that meet at a vertex situated at the expected value of the median for an odd sample size. These vertices are connected by a nearly vertical line, and it is interesting to note how rapidly they approach the population median $-\ln(-\ln 0.50)=0.3665\ldots$ as sample size n increases.

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[1] Figures in brackets indicate the literature references at the end of this paper.

[2] The scale parameter β is the same as the parameter $1/\alpha$ used in reference [1].

[3] This section has been contributed by B. F. Kimball.

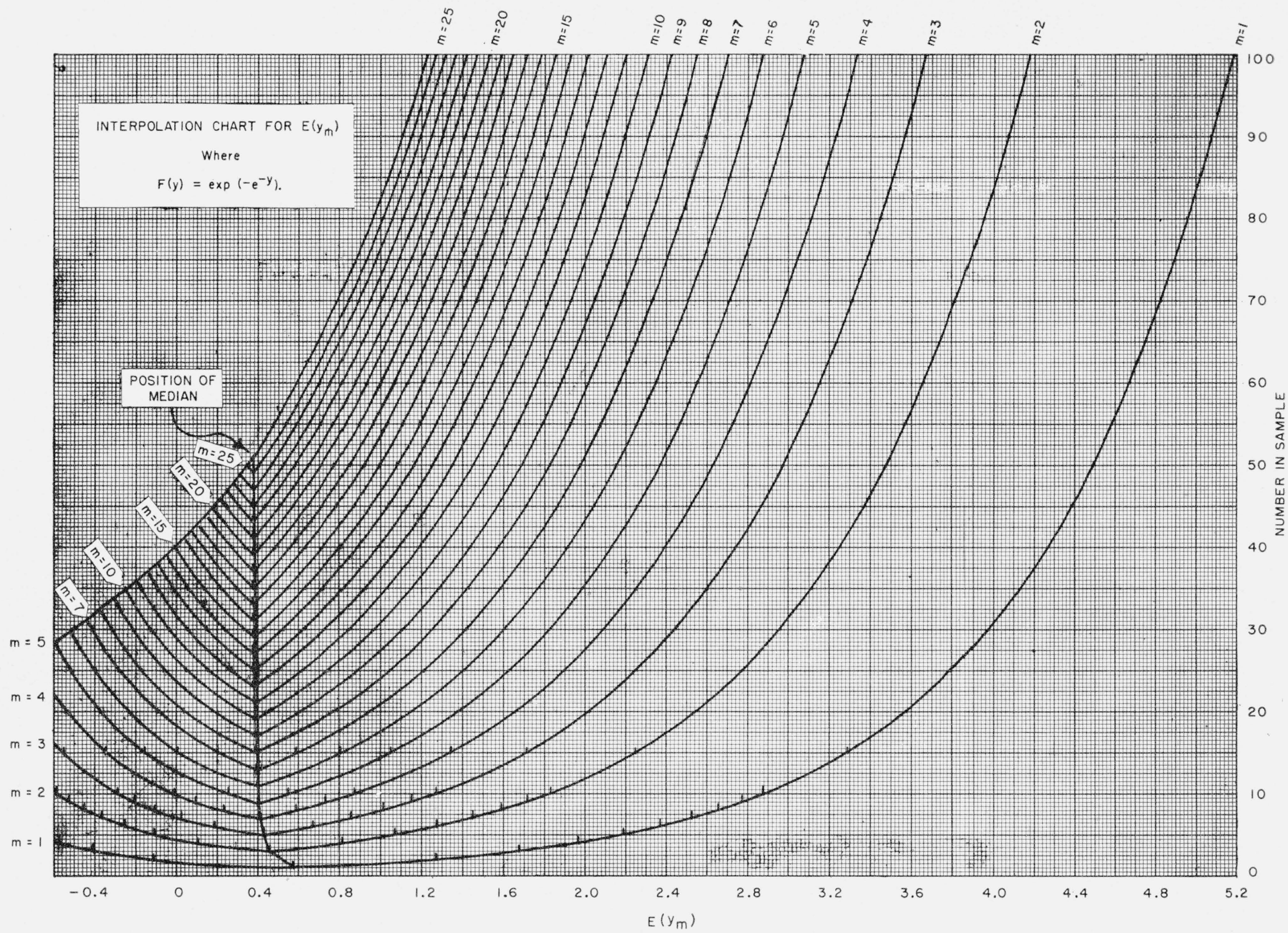

FIGURE 1.

The curves for m=constant rising to the left do not necessarily terminate in the positions shown in this figure. However, values in the range beyond the curves as shown are rarely needed. In other words, the line joining these apparent end-points might better have been omitted.

For samples of n not in the table, the interpolation chart may be used mainly for finding the expected values for the ranks in the upper half of the sample. For example, let $n=75$. Then the intersections of the horizontal line $n=75$ with the curves $m=1$, $m=2, \ldots$, are read from the horizontal scale at the bottom as follows:

$n=75$	
m	$E(y_m)$
1	4.90
2	3.88
3	3.38
4	3.04
.	.
.	.
.	.
26	0.88

3. Method of Computation

Logarithms of N from 1 to 120 to 48 decimal places from J. Wolfram's table [6] were punched on cards. The successive differences of the logarithms were then obtained with the aid of the IBM CPC–I. Sufficient differences were taken to attest to the accuracy of the functions and the individual differences. Values of the binomial coefficients were obtained from Peters and Stein's table [7], and the computation of $E(y_m)$ from eq (1) was then performed using desk calculators. The following partial check was then applied for $n \leq 25$:

$$\frac{1}{n}\sum_{m=1}^{n} E(y_m)=C.$$

For $n>25$ a differencing check was applied. The results are believed to be correct to within 2 units in the last significant figure.

TABLE 1

$$E(y_m)=C+\sum_{t=0}^{m-1}(-1)^t \binom{n}{t} \Delta^t \ln (n-t)., \text{ where } C=\text{Euler's constant}=0.57721566\ \ldots$$

Rank from top m	$n=1$	$n=2$	$n=3$	Rank from top m
1	0.57721 566	+1.2703 628	1.6758 280	1
2		−0.1159 315	+0.4594 326	2
3			−0.4036 136	3
m	$n=4$	$n=5$	$n=6$	m
1	1.9635 100	2.1866 536	2.3689 751	1
2	0.8127 817	1.0709 358	1.2750 458	2
3	+.1060 835	+0.4255 506	0.6627 159	3
4	−.5735 126	−.1068 945	+.1883 853	4
5		−.6901 671	−.2545 345	5
6			−.7772 937	6
m	$n=7$	$n=8$	$n=9$	m
1	2.5231 258	2.6566 572	2.7744 402	1
2	1.4440 711	1.5884 061	1.7143 929	2
3	0.8524 826	1.0110 660	1.1474 521	3
4	.4096 935	0.5881 770	0.7382 939	4
5	+.0224 042	+.2312 101	.4005 308	5
6	−.3653 099	−.1028 793	+.0957 534	6
7	−.8459 576	−.4527 868	−.2021 957	7
8		−.9021 249	−.5243 843	8
9			−.9493 425	9
m	$n=10$	$n=15$	$n=20$	m
1	2.8798 008	3.2852 659	3.5729 479	1
2	1.8261 956	2.2503 728	2.5470 821	2
3	1.2671 822	1.7132 872	2.0200 359	3
4	0.8689 818	1.3403 534	1.6583 531	4
5	.5436 122	1.0478 353	1.3785 557	5
6	+.2574 495	0.8018 873	1.1471 396	6
7	−.0120 439	.5851 754	0.9472 338	7
8	−.2836 893	.3872 821	.7690 671	8
9	−.5845 581	.2010 161	.6063 833	9
10	−.9898 741	+.0206 140	.4548 347	10
11		−.1594 579	.3111 607	11
12		−.3458 052	.1727 124	12
13		−.5485 327	+.0371 321	13
14		−.7883 747	−.0979 135	14
15		−1.1326 841	−.2350 333	15
16			−.3775 607	16
17			−.5304 220	17
18			−.7022 321	18
19			−.9119 529	19
20			−1.2232 104	20
m	$n=25$	$n=30$	$n=35$	m
1	3.7960 915	3.9784 130	4.1325 637	1
2	2.7755 416	2.9613 665	3.1179 999	2
3	2.2542 557	2.4438 174	2.6030 713	3
4	1.8988 407	2.0923 991	2.2544 241	4
5	1.6258 959	1.8237 416	1.9887 029	5
6	1.4020 114	1.6044 708	1.7725 500	6
7	1.2104 327	1.4178 745	1.5892 723	7
8	1.0415 346	1.2543 779	1.4293 167	8
9	0.8892 472	1.1079 713	1.2866 988	9
10	.7494 644	0.9746 216	1.1574 147	10
11	.6192 453	.8514 772	1.0386 471	11
12	.4963 746	.7364 333	0.9283 319	12
13	.3791 010	.6278 773	.8249 044	13
14	.2659 691	.5245 305	.7271 437	14
15	.1557 008	.4253 445	.6340 712	15
16	+.0471 025	.3294 297	.5448 828	16
17	−.0610 213	.2360 031	.4589 015	17
18	−.1699 655	.1443 471	.3755 427	18
19	−.2812 545	+.0537 740	.2942 883	19
20	−.3968 504	−.0364 087	.2146 659	20
21	−.5195 330	−.1269 375	.1362 310	21
22	−.6536 981	−.2186 413	+.0585 513	22
23	−.8073 485	−.3125 115	−.0188 098	23
24	−.9984 861	−.4098 117	−.0963 110	24
25	−1.2882 598	−.5122 664	−.1744 547	25
26		−.6224 196	−.2538 157	26
m	$n=40$	$n=45$	$n=50$	m
1	4.2660 951	4.3838 782	4.4892 387	1
2	3.2533 828	3.3725 996	3.4791 033	2
3	2.7403 936	2.8611 039	2.9687 929	3
4	2.3937 809	2.5160 483	2.6249 676	4
5	2.1301 969	2.2540 897	2.3642 868	5

Rank from top m	$n=40$	$n=45$	$n=50$	Rank from top m
6	1.9162 926	2.0418 841	2.1534 099	6
7	1.7353 848	1.8627 537	1.9756 620	7
8	1.5779 311	1.7071 623	1.8215 109	8
9	1.4379 596	1.5691 446	1.6849 953	9
10	1.3114 804	1.4447 183	1.5621 370	10
11	1.1956 921	1.3310 904	1.4501 482	11
12	1.0885 487	1,2262 243	1.3469 974	12
13	0.9885 067	1.1285 871	1.2511 576	13
14	.8943 693	1.0369 939	1.1614 507	14
15	.8051 863	0.9505 080	1.0769 470	15
16	.7201 865	.8683 738	0.9968 991	16
17	.6387 319	.7899 709	.9206 958	17
18	.5602 845	.7147 813	.8478 294	18
19	.4843 816	.6423 661	.7778 727	19
20	.4106 182	.5723 476	.7104 609	20
21	.3386 325	.5043 959	.6452 792	21
22	.2680 942	.4382 186	.5820 524	22
23	.1986 951	.3735 525	.5205 370	23
24	.1301 405	.3101 567	.4605 148	24
25	+.0621 408	.2478 069	.4017 883	25
26	−.0055 967	.1862 904	.3441 759	26
m	$n=55$	$n=60$	$n=70$	m
1	4.5845 489	4.6715 602	4.8257 109	1
2	3.5753 462	3.6631 331	3.8184 993	2
3	3.0659 996	3.1545 855	3.3111 996	3
4	2.7231 708	2.8125 802	2.9704 763	4
5	2.4635 211	2.5537 800	2.7129 934	5
6	2.2537 117	2.3448 473	2.5054 151	6
7	2.0770 701	2.1691 110	2.3310 720	7
8	1.9240 664	2.0170 428	2.1804 376	8
9	1.7887 415	1.8826 854	2.0475 566	9
10	1.6671 204	1.7620 653	1.9284 576	10
11	1.5564 180	1.6523 995	1.8203 598	11
12	1.4546 060	1.5516 619	1.7212 394	12
13	1.3601 612	1.4583 314	1.6295 780	13
14	1.2719 091	1.3712 360	1.5442 063	14
15	1.1889 245	1.2894 532	1.4642 047	15
16	1.1104 647	1.2122 431	1.3888 367	16
17	1.0359 236	1.1390 029	1.3175 026	17
18	0.9647 993	1.0692 342	1.2497 078	18
19	.8966 708	1.0025 197	1.1850 392	19
20	.8311 807	0.9385 062	1.1231 476	20
21	.7680 221	.8768 913	1.0637 356	21
22	.7069 285	.8174 137	1.0065 469	22
23	.6476 664	.7598 456	0.9513 592	23
24	.5900 293	.7039 865	.8979 781	24
25	.5338 321	.6496 584	.8462 324	25
26	.4789 079	.5967 022	.7959 700	26
m	$n=80$	$n=90$	$n=100$	m
1	4.9592 423	5.0770 253	5.1823 859	1
2	3.9529 397	4.0714 283	4.1773 523	2
3	3.4465 700	3.5657 784	3.6722 760	3
4	3.1067 984	3.2267 412	3.3338 229	4
5	2.8502 897	2.9709 823	3.0786 589	5
6	2.6437 091	2.7651 672	2.8734 497	6
7	2.4703 881	2.5926 280	2.7015 279	7
8	2.3208 011	2.4438 397	2.5533 687	8
9	2.1889 938	2.3128 486	2.4230 189	9
10	2.0709 961	2.1956 850	2.3065 092	10
11	1.9640 281	2.0895 700	2.2010 610	11
12	1.8660 674	1.9924 818	2.1046 529	12
13	1.7755 969	1.9029 038	2.0157 689	13
14	1.6914 486	1.8196 690	1.9332 423	14
15	1.6127 044	1.7418 600	1.8561 563	15
16	1.5386 292	1.6687 427	1.7837 772	16
17	1.4686 252	1.5997 200	1.7155 086	17
18	1.4021 995	1.5343 001	1.6508 589	18
19	1.3389 408	1.4720 725	1.5894 186	19
20	1.2785 021	1.4126 915	1.5308 422	20
21	1.2205 878	1.3558 625	1.4748 362	21
22	1.1649 442	1.3013 329	1.4211 483	22
23	1.1113 514	1.2488 842	1.3695 608	23
24	1.0596 175	1.1983 258	1.3198 839	24
25	1.0095 742	1.1494 908	1.2719 514	25
26	0.9610 726	1.1022 318	1.2256 168	26

The calculations were done in the Computation Laboratory of the National Bureau of Standards. Computations were performed under the supervision of Herbert E. Salzer. The final checking was carried out under the direction of Irene Stegun.

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